Sound propagation in slowly varying lined flow ducts of arbitrary cross-section

By S. W. RIENSTRA

Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
s.w.rienstra@tue.nl

Sound transmission through ducts of constant cross-section with a uniform inviscid mean flow and a constant acoustic lining (impedance wall) is classically described by a modal expansion, where the modes are eigenfunctions of the corresponding Laplace eigenvalue problem along a duct cross-section. A natural extension for ducts with cross-section and wall impedance that are varying slowly (compared to a typical acoustic wavelength and a typical duct radius) in the axial direction is a multiple-scales solution. This has been done for the simpler problem of circular ducts with homentropic irrotational flow. In the present paper, this solution is generalized to the problem of ducts of arbitrary cross-section. It is shown that the multiple-scales problem allows an exact solution, given the cross-sectional Laplace eigensolutions. The formulation includes both hollow and annular geometries. In addition, the turning point analysis is given for a single hard-wall cut-on, cut-off transition. This appears to yield the same reflection and transmission coefficients as in the circular duct problem.

1. Introduction

The sound field in a duct of constant cross-section with linear boundary conditions that are independent of the axial coordinate may be described by an infinite sum of modes, consisting of the eigenfunctions of the Laplace operator corresponding to a duct cross-section. Consider the two-dimensional area $\mathcal{A}$ with a smooth boundary $\partial\mathcal{A}$ and an externally directed unit normal $\boldsymbol{n}$ (figure 1). For physical relevance $\mathcal{A}$ should be simply connected, otherwise we would have several independent ducts. When we consider, for definiteness, this area be part of the (y, z)-plane, it describes the duct $\mathscr{D}$ given by

$$\mathscr{D} = \{\boldsymbol{x} = (x, y, z) | (0, y, z) \in \mathcal{A}\}$$

with axial cross-sections being copies of $\mathcal{A}$ and the normal vectors $\boldsymbol{n}$ are the same for all x. Assume in the duct a field ϕ satisfying the reduced wave equation with boundary conditions

$$\nabla^2\phi + \omega^2\phi = 0 \quad \text{for} \quad \boldsymbol{x} \in \mathscr{D}, \quad \text{with} \quad \mathscr{B}(\phi) = 0 \quad \text{for} \quad \boldsymbol{x} \in \partial\mathscr{D},$$

where $\mathscr{B}$ is typically of the form

$$\mathscr{B}(\phi) = a(y, z)\boldsymbol{n} \cdot \nabla\phi + b(y, z)\phi + c(y, z)\frac{\partial}{\partial x}\phi,$$

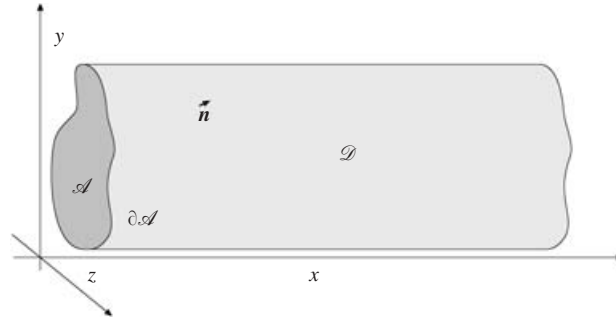although more derivatives with respect to x would not essentially alter the result.

FIGURE 1. Straight duct.

The solution of this problem may be given by

$$\phi(x, y, z) = \sum_{n=0}^{\infty} C_n \psi_n(y, z) \, \mathrm{e}^{-\mathrm{i} k_n x}$$

where ψ_n are the eigenfunctions of the Laplace operator reduced to $\mathscr{A}$, i.e. solutions of

$$-\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \alpha^2 \psi \quad \text{for} \quad (y, z) \in \mathscr{A}, \quad \text{with} \quad \tilde{\mathscr{B}}(\psi; \alpha) = 0 \quad \text{for} \quad (y, z) \in \partial\mathscr{A},$$

and α^2 is the corresponding eigenvalue. The axial wavenumber k is defined through the dispersion relation $k^2 = \omega^2 - \alpha^2$ and the reduced boundary condition operator $\tilde{\mathscr{B}}$ is given by

$$\tilde{\mathscr{B}}(\psi; \alpha) = a(y, z)\boldsymbol{n} \cdot \nabla\psi + b(y, z)\psi - \mathrm{i} k(\alpha)c(y, z)\psi.$$

If the duct cross-section is circular or rectangular while the boundary condition is uniform everywhere, the solutions of the eigenvalue problem are relatively simple: combinations of exponentials and Besselfunctions in the circular case and combinations of trigonometric functions in the rectangular case. Some other geometries, like ellipses, also allow explicit solutions. For other geometries one has to fall back on numerical methods for the eigenvalue problem. If the duct contains a uniform mean flow, the above solution is only little different.

Each term in the series expansion, i.e. $\psi_n(y, z)\mathrm{e}^{-\mathrm{i} k_n x}$, is called a duct mode. Mathematically, these modes are interesting because (in general) they form a complete basis by which any other solution can be represented. Physically, they are interesting because the usually complicated behaviour of the total field is more easily understood via the simple properties of the elements.

A very important application of a problem of this kind is the sound propagation in the inlet or bypass duct of a turbofan aircraft engine. Near the fan, the duct cross-section is necessarily circularly annular, but in other parts both diameter and shape may vary. Because of the mean flow, these variations are necessarily gradual and slow. By utilizing this slow variation the above analysis was extended in Rienstra (1999) to slowly varying modes, by way of an application of the method of multiple scales or the WKB method, for an isentropic irrotational mean flow in a circular, hollow and annular, lined duct. In Rienstra & Eversman (2001) this solution was compared with a numerically 'exact' solution. Cooper and Peake expanded this result extensively in two ways. In Peake & Cooper (2001), the solution was given for hard-walled ducts of cross-sections of elliptic shape with irrotational mean flow. In Cooper & Peake (2001)

the analysis was given for lined circular (hollow and annular) ducts with mean flow with *swirl*. This last case is far more complicated than the one for irrotational flow, as the equations are not self-adjoint.

In the present paper we will generalize the solutions for irrotational isentropic mean flow to that of slowly varying modes in ducts of arbitrary cross-section and arbitrary boundary conditions of impedance type. The paper is structured as follows. We start, in §2, by formulating the model considered of irrotational isentropic mean flow with perturbations in non-dimensional form, the geometry with slow variation in the axial direction, and the boundary conditions. Then, in §3, the mean flow is solved asymptotically to leading order, as far as is necessary for the final result. Section 4, which is the major part of the paper, is devoted to the asymptotic solution of the acoustic field in the form of a slowly varying duct mode. Then in §5 it is shown that the solution found incorporates the previously found solutions. Finally in §6, we consider the turning point problem of a hard-walled mode passing a cut-on/cut-off transition. This is of interest because here the asymptotic theory breaks down.

2. The problem

2.1. *The equations*

In the acoustic realm of a perfect gas that we will consider (infinite Reynolds and Péclet numbers, constant heat capacities), we have for pressure $\tilde{p}$, velocity $\tilde{v}$, density $\tilde{\rho}$, entropy $\tilde{s}$, and sound speed $\tilde{c}$

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho} = -\tilde{\rho}\nabla\cdot\tilde{v}, \quad \tilde{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{v} = -\nabla\tilde{p}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\tilde{s} = 0, \\ \tilde{s} = C_V \log \tilde{p} - C_P \log \tilde{\rho}, \quad \tilde{c}^2 = \frac{\gamma\tilde{p}}{\tilde{\rho}}, \quad \gamma = \frac{C_P}{C_V}, \end{aligned}\right\} \tag{1}$$

where γ, C_P and C_V are gas constants. C_V is the heat capacity at constant volume, C_P is the heat capacity at constant pressure, and $\gamma = C_P/C_V$. We have a stationary mean flow with unsteady time-harmonic perturbations of frequency $\omega > 0$, given, in the usual complex notation, by

$$\tilde{v} = V + \mathrm{Re}(v\,\mathrm{e}^{\mathrm{i}\omega t}), \quad \tilde{p} = P + \mathrm{Re}(p\,\mathrm{e}^{\mathrm{i}\omega t}), \quad \tilde{\rho} = D + \mathrm{Re}(\rho\,\mathrm{e}^{\mathrm{i}\omega t}), \quad \tilde{s} = S + \mathrm{Re}(s\,\mathrm{e}^{\mathrm{i}\omega t}). \tag{2}$$

For notational convenience we introduced capital letters to denote the mean flow part. To avoid confusion we use for density D rather than P. When we linearize for small amplitude, we obtain for the mean flow

$$\left.\begin{aligned} \nabla\cdot(DV) = 0, \quad D(V\cdot\nabla)V = -\nabla P, \quad (V\cdot\nabla)S = 0, \\ S = C_V \log P - C_P \log D, \quad C^2 = \frac{\gamma P}{D} \end{aligned}\right\} \tag{3}$$

and for the perturbations

$$\mathrm{i}\omega\rho + \nabla\cdot(V\rho + vD) = 0, \tag{4a}$$

$$D(\mathrm{i}\omega + V\cdot\nabla)v + D(v\cdot\nabla)V + \rho(V\cdot\nabla)V = -\nabla p, \tag{4b}$$

$$(\mathrm{i}\omega + V\cdot\nabla)s + v\cdot\nabla S = 0, \tag{4c}$$

with

$$s = \frac{C_V}{P}p - \frac{C_P}{D}\rho = \frac{C_V}{P}(p - C^2\rho). \tag{4d}$$
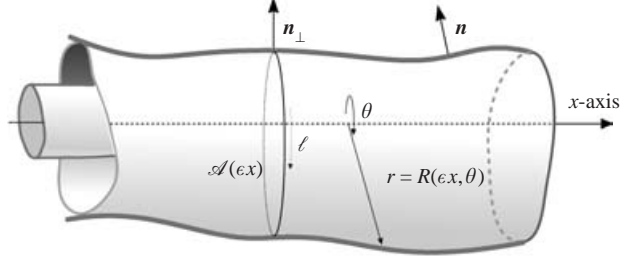
FIGURE 2. Sketch of the geometry.

Assuming that the flow field $\tilde{\boldsymbol{v}}$ is irrotational and isentropic everywhere (homentropic), we can introduce a potential for the velocity, where $\tilde{\boldsymbol{v}} = \nabla\tilde{\phi}$ and $\tilde{\phi} = H + \mathrm{Re}(\phi\,\mathrm{e}^{i\omega t})$, and express $\tilde{p}$ as a function of $\tilde{\rho}$ only, such that we can integrate the momentum equation (Bernoulli's law, with constant E), to obtain for the mean flow

$$\tfrac{1}{2}V^2 + \frac{C^2}{\gamma - 1} = E, \quad \nabla\cdot(D\boldsymbol{V}) = 0, \quad \frac{P}{D^\gamma} = \text{constant}, \quad C^2 = \gamma\frac{P}{D} \tag{5}$$

(where $V = |\boldsymbol{V}|$) and for the acoustic perturbations

$$(\mathrm{i}\omega + \boldsymbol{V}\cdot\nabla)\rho + \rho\nabla\cdot\boldsymbol{V} + \nabla\cdot(D\nabla\phi) = 0, \quad D(\mathrm{i}\omega + \boldsymbol{V}\cdot\nabla)\phi + p = 0, \quad p = C^2\rho. \tag{6}$$

These last equations are further simplified (eliminate p and ρ and use the fact that $\nabla\cdot(D\boldsymbol{V}) = 0$) to the quite general convected reduced wave equation

$$D^{-1}\nabla\cdot(D\nabla\phi) - (\mathrm{i}\omega + \boldsymbol{V}\cdot\nabla)[C^{-2}(\mathrm{i}\omega + \boldsymbol{V}\cdot\nabla)\phi] = 0. \tag{7}$$

2.2. *Non-dimensionalization*

Without further change of notation, we will assume throughout this paper that the problem is made dimensionless: lengths on a typical duct radius, time on typical sound speed/typical duct radius, pressure on typical density $\times$ (sound speed)^2.

2.3. *The geometry*

The domain of interest consists of a duct $\mathscr{V}$ of arbitrary cross-section, slowly varying in the axial direction (see figure 2). For definiteness, it is given by the function Σ in cylindrical coordinates (x, r, θ) as follows:

$$\Sigma(X, r, \theta) = r - R(X, \theta) \leqslant 0 \tag{8}$$

where $X = \varepsilon x \geqslant 0$ is a so-called slow variable and ε is small. A cross-section $\mathscr{A}(X)$ at X has surface area $A(X)$. In order to avoid unnecessary complexity in notation, $\Sigma = 0$ corresponds to the surface of a hollow cylinder, but the analysis is easily generalized to topologically more complex shapes. The results finally presented will be valid for any hollow or annular ducts.

At the duct surface $\Sigma = 0$ the gradient $\nabla\Sigma$

$$\nabla\Sigma = -\varepsilon\boldsymbol{e}_x R_X + \boldsymbol{e}_r - \boldsymbol{e}_\theta\frac{1}{R}R_\theta, \quad \text{with} \quad \nabla = \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} \tag{9}$$

(where an index denotes a partial derivative, except for $\boldsymbol{e}_x, \boldsymbol{e}_r, \boldsymbol{e}_\theta$ which denote unit basis vectors) is a vector normal to the surface, so

$$\boldsymbol{n} = \frac{\nabla\Sigma}{|\nabla\Sigma|}, \tag{10}$$

while the transverse gradient $\nabla_\perp \Sigma$

$$\nabla_\perp \Sigma = e_r - e_\theta \frac{1}{R} R_\theta, \quad \text{with} \quad \nabla_\perp = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta}, \tag{11}$$

is directed in the plane of a cross-section $\mathscr{A}(X)$, and normal to the perimeter $\partial \mathscr{A}$. So if $n_\perp$ denotes the component of the surface-normal vector n in the plane of a cross-section, we have

$$n_\perp = \frac{\nabla_\perp \Sigma}{|\nabla_\perp \Sigma|}. \tag{12}$$

Note that $n_\perp$, written in terms of X, is independent of ε, and

$$n = n_\perp - \varepsilon \frac{R R_X}{\sqrt{R^2 + R_\theta^2}} e_x + O(\varepsilon^2). \tag{13}$$

The orientation of θ and perimeter arclength ℓ are chosen as indicated in figure 2, i.e. such that

$$e_\ell = e_x \times n_\perp, \qquad e_\theta = e_x \times e_r. \tag{14}$$

2.4. *Boundary conditions*

The duct wall is impermeable to the mean flow, so we have the mean flow boundary condition

$$V \cdot n = 0 \quad \text{at} \quad \Sigma = 0. \tag{15}$$

If we denote the mean flow by $V = U e_x + V_\perp$, with the axial component U and the cross-wise component $V_\perp$, the mean flow mass flux, given by

$$\iint_{\mathscr{A}} D U \, d\sigma = \mathscr{F}, \tag{16}$$

is independent of x. The mean flow is assumed to be determined by the slowly varying geometry only. For example, entrance effects that scale on a duct diameter are not present. The acoustic boundary condition of an impedance wall along a curved wall with mean flow is, according to Myers (1980), given by

$$i\omega(v \cdot n) = [i\omega + V \cdot \nabla - n \cdot (n \cdot \nabla V)]\left(\frac{p}{Z}\right) \quad \text{at} \quad \Sigma = 0. \tag{17}$$

The impedance Z may vary with position, provided it varies slowly in the x-direction, so $Z = Z(X, \theta)$.

3. Mean flow

We assume the mean flow to be determined by the slowly varying geometry only. For definiteness, we will assume in particular that far upstream the duct is of uniform cross-section and the mean flow is independent of ε. We will apply the method of slow variation (Van Dyke 1987) to construct an approximate solution. We write all flow variables as a function of $(X, r, \theta; \varepsilon)$. Rewritten in terms of X, the equations are now dependent only on ε^2, while the upstream conditions are independent of ε. It follows that the flow is now a function only of ε^2, and – assuming regularity in this parameter – we expand each variable in a regular Poincaré expansion in powers of ε^2.

From elementary order of magnitude considerations it follows that $U = O(1)$, $V_\perp = O(\varepsilon)$, $H = O(\varepsilon^{-1})$, $D = O(1)$, $C = O(1)$, and $P = O(1)$. So we have

$$H = \varepsilon^{-1}H_0 + \varepsilon H_1 + O(\varepsilon^3), \quad U = U_0 + O(\varepsilon^2), \quad V_\perp = \varepsilon V_{\perp 0} + O(\varepsilon^3),$$
$$D = D_0 + O(\varepsilon^2), \qquad\qquad P = P_0 + O(\varepsilon^2), \quad C = C_0 + O(\varepsilon^2), \tag{18}$$

where each term in the expansion is independent of ε. We substitute these expansions in the equation of mass conservation and in the boundary conditions at $\Sigma = 0$ and collect the terms of like powers of ε. Then we obtain to leading order

$$\nabla_\perp \cdot (D_0 \nabla_\perp H_0) = 0, \quad \text{with} \quad \nabla_\perp H_0 \cdot \boldsymbol{n}_\perp = 0 \quad \text{at} \quad r = R. \tag{19}$$

A solution for H_0 is evidently $\nabla_\perp H_0 \equiv \boldsymbol{0}$, in other words $H_0 = H_0(X)$. Moreover, this solution is indeed unique, as may be seen from the following integral along a cross-section $\mathscr{A}$:

$$\iint_\mathscr{A} H_0 \nabla_\perp \cdot (D_0 \nabla_\perp H_0)\, \mathrm{d}\sigma = \iint_\mathscr{A} \nabla_\perp \cdot (H_0 D_0 \nabla_\perp H_0) - D_0 |\nabla_\perp H_0|^2 \,\mathrm{d}\sigma$$
$$= \int_{\partial\mathscr{A}} H_0 D_0 (\nabla_\perp H_0 \cdot \boldsymbol{n}_\perp)\, \mathrm{d}\ell - \iint_\mathscr{A} D_0 |\nabla_\perp H_0|^2 \,\mathrm{d}\sigma = -\iint_\mathscr{A} D_0 |\nabla_\perp H_0|^2 \,\mathrm{d}\sigma = 0. \tag{20}$$

So for any $D_0 > 0$, $|\nabla_\perp H_0|^2 \equiv 0$.

From the leading order of Bernoulli's equation and the relations between P, D and C

$$\tfrac{1}{2}U_0^2 + \frac{C_0^2}{\gamma - 1} = E, \quad C_0^2 = \frac{\gamma P_0}{D_0}, \quad \frac{P_0}{D_0^\gamma} = \gamma^{-1}, \tag{21}$$

we find that C_0 and thus D_0 and P_0 are also a function of X only. U_0 is found from the given mass flux $\mathscr{F}$ through a cross-section $\mathscr{A}$ with surface $A(X)$:

$$U_0(X) = \frac{\mathscr{F}}{D_0(X)A(X)}, \tag{22}$$

and D_0 (and hence C_0 and P_0) is found as the root of the algebraic equation that results from Bernoulli's equation

$$\frac{\mathscr{F}^2}{2D_0^2 A^2} + \frac{D_0^{\gamma-1}}{\gamma - 1} = E. \tag{23}$$

Altogether we have a nearly uniform mean flow

$$\boldsymbol{V}(X, r, \theta; \varepsilon) = U_0(X)\boldsymbol{e}_x + \varepsilon \boldsymbol{V}_{\perp 0}(X, r, \theta) + O(\varepsilon^2),$$
$$D(X, r, \theta; \varepsilon) = D_0(X) + O(\varepsilon^2), \quad C(X, r, \theta; \varepsilon) = C_0(X) + O(\varepsilon^2). \tag{24}$$

The mean flow cross-wise component $\boldsymbol{V}_{\perp 0}$ is defined by

$$\frac{\partial}{\partial X}(D_0 U_0) + \nabla_\perp \cdot (D_0 \boldsymbol{V}_{\perp 0}) = 0, \quad \text{with} \quad \boldsymbol{V}_{\perp 0} \cdot \boldsymbol{n}_\perp = \frac{R R_X}{\sqrt{R^2 + R_\theta^2}} U_0 \quad \text{at} \quad r = R, \tag{25}$$

but it is not determined here, as it does not appear in the final result.

We finally note that the operator that occurs in the acoustic boundary condition becomes

$$\mathrm{i}\omega + \boldsymbol{V} \cdot \nabla - \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{V}) = \mathrm{i}\omega + U_0 \frac{\partial}{\partial x} + \varepsilon(\boldsymbol{V}_{\perp 0} \cdot \nabla_\perp - \boldsymbol{n}_\perp \cdot (\boldsymbol{n}_\perp \cdot \nabla_\perp \boldsymbol{V}_{\perp 0})) + O(\varepsilon^2). \tag{26}$$

4. Acoustic field

The equation for the acoustic field ϕ becomes, under the above approximation,

$$
\begin{aligned}
&\phi_{xx} + \nabla_\perp^2 \phi - C_0^{-2}\left[-\omega^2\phi + 2\mathrm{i}\omega U_0\phi_x + U_0^2\phi_{xx}\right] \\
&+ \varepsilon\left\{D_0^{-1}D_{0,X}\phi_x - \mathrm{i}\omega U_0\left(C_0^{-2}\right)_X\phi - U_0\left(U_0C_0^{-2}\right)_X\phi_x\right. \\
&\left. - 2\mathrm{i}\omega C_0^{-2}(\boldsymbol{V}_{\perp0}\cdot\nabla_\perp\phi) - 2U_0C_0^{-2}(\boldsymbol{V}_{\perp0}\cdot\nabla_\perp\phi_x)\right\} + O(\varepsilon^2) = 0.
\end{aligned}
\tag{27}
$$

The assumption of a multiple-scales solution is equivalent here to the WKB-Ansatz:

$$
\phi = \Phi(X, r, \theta; \varepsilon)\exp\left(-\mathrm{i}\int^x \mu(\varepsilon\xi; \varepsilon)\,\mathrm{d}\xi\right),
\tag{28a}
$$

$$
\phi_x = (-\mathrm{i}\mu\Phi + \varepsilon\Phi_X)\exp\left(-\mathrm{i}\int^x \mu(\varepsilon\xi; \varepsilon)\,\mathrm{d}\xi\right),
\tag{28b}
$$

$$
\phi_{xx} = (-\mu^2\Phi - \mathrm{i}\varepsilon\mu_X\Phi - 2\mathrm{i}\varepsilon\mu\Phi_X + \varepsilon^2\Phi_{XX})\exp\left(-\mathrm{i}\int^x \mu(\varepsilon\xi; \varepsilon)\,\mathrm{d}\xi\right),
\tag{28c}
$$

$$
p = -D_0(\mathrm{i}\Omega\Phi + \varepsilon U_0\Phi_X + \varepsilon\boldsymbol{V}_{\perp0}\cdot\nabla_\perp\Phi)\exp\left(-\mathrm{i}\int^x \mu(\varepsilon\xi; \varepsilon)\,\mathrm{d}\xi\right).
\tag{28d}
$$

Introduce

$$
\Omega = \omega - \mu U_0,
\tag{29}
$$

and substitute the above to obtain after some simplifications

$$
\nabla_\perp^2\Phi + \left(\frac{\Omega^2}{C_0^2} - \mu^2\right)\Phi = \frac{\mathrm{i}\varepsilon}{D_0\Phi}\left[\left[\left(\frac{\Omega U_0}{C_0^2} + \mu\right)D_0\Phi^2\right]_X + \nabla_\perp\cdot\left(\frac{\Omega D_0}{C_0^2}\Phi^2\boldsymbol{V}_{\perp0}\right)\right] + O(\varepsilon^2),
\tag{30a}
$$

where use is made of $-(D_0U_0)_X = \nabla_\perp\cdot(D_0\boldsymbol{V}_{\perp0})$. We obtain for the boundary condition at $r = R$

$$
\begin{aligned}
&\mathrm{i}\omega(\boldsymbol{n}_\perp\cdot\nabla_\perp\Phi) - \frac{\Omega^2 D_0}{Z}\Phi \\
&= \varepsilon\omega\mu\frac{RR_X}{\sqrt{R^2 + R_\theta^2}}\Phi - \mathrm{i}\varepsilon\left[U_0\left(\frac{D_0\Omega\Phi}{Z}\right)_X + U_0\frac{D_0\Omega}{Z}\Phi_X\right. \\
&\left. + D_0\Omega\boldsymbol{V}_{\perp0}\cdot\nabla_\perp\left(\frac{\Phi}{Z}\right) + \frac{D_0\Omega}{Z}\boldsymbol{V}_{\perp0}\cdot\nabla_\perp\Phi\right] + \mathrm{i}\varepsilon\boldsymbol{n}_\perp\cdot(\boldsymbol{n}_\perp\cdot\nabla_\perp\boldsymbol{V}_{\perp0})\frac{D_0\Omega\Phi}{Z}.
\end{aligned}
\tag{30b}
$$

We expand in the usual way:

$$
\Phi(X, r, \theta; \varepsilon) = \Phi_0(X, r, \theta) + \varepsilon\Phi_1(X, r, \theta) + O(\varepsilon^2), \quad \mu(X; \varepsilon) = \mu(X) + O(\varepsilon^2),
\tag{31}
$$

where we ignored for notational convenience the subscript of μ.

To leading order we obtain

$$
\nabla_\perp^2\Phi_0 + \left(\frac{\Omega^2}{C_0^2} - \mu^2\right)\Phi_0 = 0, \quad \text{with} \quad \mathrm{i}\omega(\boldsymbol{n}_\perp\cdot\nabla_\perp\Phi_0) - \frac{\Omega^2 D_0}{Z}\Phi_0 = 0 \quad \text{at} \quad r = R.
\tag{32}
$$

This is formally solved by the solution of the following eigenvalue problem in a cross-sectional plane $\mathscr{A}$, with X acting as a parameter:

$$
-\nabla_\perp^2\psi = \alpha^2\psi, \quad \text{with} \quad (\boldsymbol{n}_\perp\cdot\nabla_\perp\psi) = \frac{\Omega^2 D_0}{\mathrm{i}\omega Z}\psi \quad \text{at} \quad r = R
\tag{33}
$$

and $\Omega = \Omega(\alpha)$ because it satisfies the dispersion relation

$$\frac{\Omega^2}{C_0^2} - \frac{(\omega - \Omega)^2}{U_0^2} = \alpha^2. \tag{34}$$

We consider the nth eigenvalue α_n^2 with eigensolution ψ_n. We assume that $\iint_{\mathscr{A}} \psi_n^2 \, d\sigma \neq 0$ so that ψ_n can be normalized as

$$\iint_{\mathscr{A}} \psi_n^2 \, d\sigma = 1. \tag{35}$$

Then we have

$$\Phi_0 = N(X)\psi_n(X, r, \theta), \quad \text{and} \quad \mu_n = \frac{\omega - \Omega(\alpha_n)}{U_0}. \tag{36}$$

The amplitude N is still unknown. This will be determined from a solvability condition (Nayfeh 1973) for the next order Φ_1. The existence of Φ_1 is not evident because we assumed the solution to be of a particular form, i.e. (28a)†. As we will not be able to determine Φ_1 entirely without considering Φ_2, we will not try to solve the full equation but restrict ourselves to a necessary condition for its solvability. This will be just enough to determine N. We have

$$\nabla_\perp^2 \Phi_1 + \alpha_n^2 \Phi_1 = \frac{i}{D_0 \Phi_0} \left[\left[\left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \Phi_0^2 \right]_X + \nabla_\perp \cdot \left(\frac{\Omega D_0}{C_0^2} \Phi_0^2 \boldsymbol{V}_{\perp 0} \right) \right], \tag{37a}$$

with at $r = R$

$$i\omega(\boldsymbol{n}_\perp \cdot \nabla_\perp \Phi_1) - \frac{\Omega^2 D_0}{Z} \Phi_1$$

$$= \omega\mu \frac{R R_X}{\sqrt{R^2 + R_\theta^2}} \Phi_0 - i \left[U_0 \left(\frac{D_0 \Omega \Phi_0}{Z} \right)_X + U_0 \frac{D_0 \Omega}{Z} \Phi_{0,X} \right.$$

$$\left. + D_0 \Omega \boldsymbol{V}_{\perp 0} \cdot \nabla_\perp \left(\frac{\Phi_0}{Z} \right) + \frac{D_0 \Omega}{Z} \boldsymbol{V}_{\perp 0} \cdot \nabla_\perp \Phi_0 \right] + i\boldsymbol{n}_\perp \cdot (\boldsymbol{n}_\perp \cdot \nabla_\perp \boldsymbol{V}_{\perp 0}) \frac{D_0 \Omega \Phi_0}{Z}. \tag{37b}$$

Multiply equation (37a) by $D_0 \Phi_0$ and equation (32) by $D_0 \Phi_1$. Integrate their difference over a cross-section $\mathscr{A}$ to obtain

$$D_0 \iint_{\mathscr{A}} \Phi_0 \nabla_\perp^2 \Phi_1 - \Phi_1 \nabla_\perp^2 \Phi_0 \, d\sigma$$

$$= i \iint_{\mathscr{A}} \left[\left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \Phi_0^2 \right]_X d\sigma + i \iint_{\mathscr{A}} \nabla_\perp \cdot \left(\frac{\Omega D_0}{C_0^2} \Phi_0^2 \boldsymbol{V}_{\perp 0} \right) d\sigma. \tag{38}$$

The first integral on the right-hand side may be recast into

$$\iint_{\mathscr{A}} \left[\left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \Phi_0^2 \right]_X d\sigma = \int_0^{2\pi} \int_0^R \frac{\partial}{\partial X} \left[\left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \Phi_0^2 \right] r \, dr \, d\theta$$

$$= \frac{d}{dX} \left[\left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \iint_{\mathscr{A}} \Phi_0^2 \, d\sigma \right] - \left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \int_0^{2\pi} \Phi_0^2 \big|_{r=R} R R_X \, d\theta. \tag{39}$$

† In the slightly more general context of the method of multiple scales, this condition is equivalent to suppression of secular terms in order to render the approximation to remain valid for scales $X = O(1)$.

The second one becomes, with equation (25),

$$\iint_{\mathscr{A}} \nabla_{\perp} \cdot \left(\frac{\Omega D_0}{C_0^2} \Phi_0^2 \boldsymbol{V}_{\perp 0} \right) d\sigma = \frac{\Omega D_0}{C_0^2} \int_{\partial\mathscr{A}} \Phi_0^2 (\boldsymbol{V}_{\perp 0} \cdot \boldsymbol{n}_{\perp}) d\ell$$

$$= \frac{\Omega D_0}{C_0^2} \int_0^{2\pi} \Phi_0^2 \big|_{r=R} (\boldsymbol{V}_{\perp 0} \cdot \boldsymbol{n}_{\perp}) \sqrt{R^2 + R_\theta^2} \, d\theta = \frac{\Omega D_0}{C_0^2} U_0 \int_0^{2\pi} \Phi_0^2 \big|_{r=R} R R_X \, d\theta. \quad (40)$$

Together these yield

$$D_0 \iint_{\mathscr{A}} \Phi_0 \nabla_{\perp}^2 \Phi_1 - \Phi_1 \nabla_{\perp}^2 \Phi_0 \, d\sigma$$

$$= i \frac{d}{dX} \left[\left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 \iint_{\mathscr{A}} \Phi_0^2 \, d\sigma \right] - i\mu D_0 \int_0^{2\pi} \Phi_0^2 \big|_{r=R} R R_X \, d\theta. \quad (41)$$

On the other hand, by using the boundary conditions for Φ_0 and Φ_1 (equations (32) and (37b)), this is also equal to

$$D_0 \int_{\partial\mathscr{A}} \Phi_0 (\boldsymbol{n}_{\perp} \cdot \nabla_{\perp} \Phi_1) - \Phi_1 (\boldsymbol{n}_{\perp} \cdot \nabla_{\perp} \Phi_0) \, d\ell$$

$$= -\frac{D_0}{\omega} \int_{\partial\mathscr{A}} \left[U_0 \Phi_0 \left(\frac{D_0 \Omega}{Z} \Phi_0 \right)_X + \frac{U_0 D_0 \Omega}{Z} \Phi_0 \Phi_{0,X} + D_0 \Omega \Phi_0 \boldsymbol{V}_{\perp 0} \cdot \nabla_{\perp} \left(\frac{\Phi_0}{Z} \right) \right.$$

$$\left. + \frac{D_0 \Omega}{Z} \Phi_0 (\boldsymbol{V}_{\perp 0} \cdot \nabla_{\perp} \Phi_0) - \boldsymbol{n}_{\perp} \cdot (\boldsymbol{n}_{\perp} \cdot \nabla_{\perp} \boldsymbol{V}_{\perp 0}) \frac{\Phi_0^2 D_0 \Omega}{Z} \right] d\ell - i\mu D_0 \int_0^{2\pi} \Phi_0^2 \big|_{r=R} R R_X \, d\theta. \quad (42)$$

Using equation (25) and the fact that $\boldsymbol{V} \cdot \boldsymbol{n} = 0$, we can combine equations (41) and (42) to obtain

$$-i\omega \frac{d}{dX} \left[\left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 N^2 \right] = \int_{\partial\mathscr{A}} \varepsilon^{-1} \mathscr{M} \left(\frac{\Omega D_0^2 \Phi_0^2}{Z} \boldsymbol{V} \right) d\ell + O(\varepsilon^2) \quad (43)$$

where we defined the operator

$$\mathscr{M}(\boldsymbol{f}) = \nabla \cdot \boldsymbol{f} - \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{f}). \quad (44)$$

Whenever $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ (as is the case here), $\mathscr{M}(\boldsymbol{f})$ happens to be just the surface divergence of $\boldsymbol{f}$ (see Appendix A). Before we continue we need an auxiliary result.

LEMMA 1. *For any sufficiently smooth vector field $\boldsymbol{f}$, with $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ at the tube surface $r = T(x, \theta)$, we have*

$$\int_{\partial\mathscr{A}} [\nabla \cdot \boldsymbol{f} - \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{f})] \sqrt{1 + \frac{T^2 T_x^2}{T^2 + T_\theta^2}} \, d\ell = \frac{d}{dx} \int_{\partial\mathscr{A}} (\boldsymbol{f} \cdot \boldsymbol{v}) \, d\ell,$$

where $\boldsymbol{v} = \boldsymbol{n} \times \boldsymbol{e}_\ell$ is the unit vector normal to $\partial\mathscr{A}$, tangential to the surface and pointing in the positive x-direction.

Proof. See Appendix A. □

Identify $T(x, \theta) = R(\varepsilon x, \theta)$, such that $1 + (R^2 R_x^2)/(R^2 + R_\theta^2) = 1 + O(\varepsilon^2)$. It is easily verified that

$$\boldsymbol{V} \cdot \boldsymbol{v} = (U \boldsymbol{e}_x + \boldsymbol{V}_{\perp}) \cdot (\boldsymbol{n} \times \boldsymbol{e}_\ell) = (\boldsymbol{n} \cdot \boldsymbol{n}_{\perp}) U - (\boldsymbol{n} \cdot \boldsymbol{e}_x)(\boldsymbol{V}_{\perp} \cdot \boldsymbol{n}_{\perp}) = U_0 + O(\varepsilon^2).$$

This yields the result

$$\int_{\partial\mathscr{A}} \varepsilon^{-1} \mathscr{M} \left(\frac{\Omega D_0^2 \Phi_0^2}{Z} V \right) d\ell = \frac{d}{dX} \int_{\partial\mathscr{A}} \frac{\Omega D_0^2}{Z} U_0 \Phi_0^2 d\ell + O(\varepsilon^2)$$

$$:= \frac{d}{dX} \left[\Omega D_0^2 U_0 N^2 \int_{\partial\mathscr{A}} \frac{1}{Z} \psi_n^2 d\ell \right].$$

Finally, we obtain now the adiabatic invariant

$$\frac{d}{dX} \left[i\omega \left(\frac{\Omega U_0}{C_0^2} + \mu \right) D_0 N^2 + D_0^2 \Omega U_0 N^2 \int_{\partial\mathscr{A}} \frac{1}{Z} \psi_n^2 d\ell \right] = 0. \tag{45}$$

It is convenient to introduce the reduced axial wavenumber

$$\sigma = \sqrt{1 - (C_0^2 - U_0^2) \frac{\alpha^2}{\omega^2}} \tag{46}$$

so that

$$\mu = \omega \frac{C_0 \sigma - U_0}{C_0^2 - U_0^2}, \quad \frac{U_0 \Omega}{C_0^2} + \mu = \frac{\omega \sigma}{C_0}, \quad \Omega = \omega C_0 \frac{C_0 - U_0 \sigma}{C_0^2 - U_0^2}. \tag{47}$$

This yields finally for the amplitude N

$$\frac{Q^2}{N^2} = \frac{\omega \sigma D_0}{C_0} + \frac{D_0^2 \Omega}{i\omega} U_0 \int_{\partial\mathscr{A}} \frac{1}{Z} \psi_n^2 d\ell \tag{48}$$

where Q^2 is an integration constant. It represents the conserved quantity, and is to be fixed at some position $X = X_0$. Of course, all along the duct, N should remain on the same branch of its (in general) complex square root. In the case of an annular duct the analysis is only a little different, and we obtain (with outer perimeter denoted by $\partial\mathscr{A}_2$ and inner perimeter by $\partial\mathscr{A}_1$)

$$\boxed{\frac{Q^2}{N^2} = \frac{\omega \sigma D_0}{C_0} + \frac{D_0^2 \Omega U_0}{i\omega} \left(\int_{\partial\mathscr{A}_2} \frac{1}{Z_2} \psi_n^2 d\ell - \int_{\partial\mathscr{A}_1} \frac{1}{Z_1} \psi_n^2 d\ell \right).} \tag{49}$$

This is the main result of this paper.

5. Special cases and previous solutions

5.1. *Hard walls*

If $Z \to \infty$, we obtain

$$\frac{\omega \sigma D_0}{C_0} N^2 = Q^2 = \text{constant}. \tag{50}$$

5.2. *No mean flow*

If $U = 0$, D and C are constant and we obtain

$$\mu N^2 = \text{constant}. \tag{51}$$

5.3. *Axisymmetric duct with constant impedance*

If $R = R(X)$, the eigenfunctions are given by

$$\psi = K(X) J_m(\alpha(X)r) \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} \quad \text{for } m \in \mathbb{N}.$$

Note that because of symmetry the eigenvalues have multiplicity 2, i.e. for every eigenvalue we have 2 eigenfunctions. K is determined by the relation

$$K^2 \int_0^{2\pi} \left\{ \begin{matrix} \cos^2(m\theta) \\ \sin^2(m\theta) \end{matrix} \right\} d\theta \int_0^R J_m(\alpha r)^2 r \, dr = \tfrac{1}{2}\pi K^2 R^2 \left(1 - \frac{m^2 - \zeta^2}{\alpha^2 R^2} \right) J_m(\alpha R)^2 = 1,$$
(52)

where $\zeta = \Omega^2 D_0 R / i\omega Z$ and π should be read as 2π if $m = 0$. There are similar expressions for an annular duct. The line integral along $\partial \mathscr{A}$ becomes (for any m)

$$\int_{\partial \mathscr{A}} \frac{1}{Z} \psi^2 \, d\ell = \frac{2}{ZR} \left(1 - \frac{m^2 - \zeta^2}{\alpha^2 R^2} \right)^{-1}.$$
(53)

The resulting expressions are equivalent to what was found in Rienstra (1999), except that there the pair $e^{im\theta}$ and $e^{-im\theta}$ was taken, instead of $\cos(m\theta)$ and $\sin(m\theta)$. In the present formulation this choice would lead to a vanishing integral over ψ^2, and it would not have been possible to normalize ψ in the way assumed here. Of course, both forms are entirely equivalent because the complex exponential is easily recovered from a suitable combination of two eigenfunctions of the same eigenvalue.

5.4. *Elliptic hard-walled duct*

The analysis of Rienstra (1999), restricted to hard walled ducts, was extended by Peake & Cooper (2001) to ducts of elliptic cross-section. The present solution includes their results, as may be seen by comparing their equation (38) (or (36))

$$M_n^2(X) = \frac{Q_0^2 C_0^2(X)}{(C_0^2 \mu + \Omega U_0) D_0 I}$$

with our equation (48) with $Z = \infty$, and noting that we normalized the eigenfunctions such that their integral I becomes equal to unity.

6. Turning point analysis

In the case of hard walls, the above analysis fails when $\sigma \to 0$. So when the medium and diameter vary in such a way that at some point $X = X_t$ wavenumber σ vanishes, the present solution breaks down. In a small interval around X_t the mode does not vary slowly and locally a different approximation is necessary. In the terminology of matched asymptotic expansions (Holmes 1995), this is a boundary layer in variable X. The analysis follows closely the circular duct case presented in Rienstra (2000), and we use a similar notation.

When σ^2 changes sign, and σ changes from real to imaginary, the mode changes from cut-on to cut-off. If X_t is isolated, such that there are no interfering neighbouring points of vanishing σ, no power is transmitted beyond X_t, and the wave has to reflect at X_t. The incident propagating mode is split up into a cut-on reflected mode and a cut-of transmitted mode (see figure 3). Therefore, a point where wavenumber σ vanishes is called a 'turning point'.

Assume at $X = X_t$ a transition from cut-on to cut-off, so

$$\sigma_t = 0, \quad \frac{d}{dX}\sigma_t^2 < 0, \quad \mu_t = 1, \quad \mu_t' > 0, \quad \frac{C_{0t}C_{0t}' - U_{0t}U_{0t}'}{C_{0t}^2 - U_{0t}^2} + \frac{\alpha_t'}{\alpha_t} > 0, \quad (54)$$

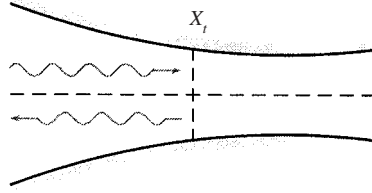where subscript t indicates evaluation at $X = X_t$ and the prime denotes a derivative with respect to X.

FIGURE 3. Turning point X_t, where a mode changes from cut-on to cut-off.

Consider incident, reflected and transmitted waves of the type found above. So in $X < X_t$, where σ is real positive, we have the incident and reflected waves

$$\phi = \frac{n(X)}{\sqrt{\sigma(X)}} \psi(r,\theta;X) \exp\left(\frac{i}{\varepsilon} \int_{X_t}^X \frac{\omega U_0}{C_0^2 - U_0^2} \, dX'\right) \left[\exp\left(-\frac{i}{\varepsilon} \int_{X_t}^X \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, dX'\right)\right.$$
$$\left. + \mathscr{R} \exp\left(\frac{i}{\varepsilon} \int_{X_t}^X \frac{\omega C_0 \sigma}{C_0^2 - U_0^2} \, dX'\right)\right] \tag{55}$$

with reflection coefficient $\mathscr{R}$ to be determined and

$$n(X) = Q\left(\frac{C_0}{\omega D_0}\right)^{1/2}. \tag{56}$$

In $X > X_t$, where σ is imaginary negative, we have the transmitted wave

$$\phi = \mathscr{T} \frac{n(X)}{\sqrt{\sigma(X)}} \psi(r,\theta;X) \exp\left(\frac{i}{\varepsilon} \int_{X_t}^X \frac{\omega U_0}{C_0^2 - U_0^2} \, dX'\right) \exp\left(-\frac{1}{\varepsilon} \int_{X_t}^X \frac{\omega C_0 |\sigma|}{C_0^2 - U_0^2} \, dX'\right) \tag{57}$$

with transmission coefficient $\mathscr{T}$ to be determined, and $\sqrt{\sigma} = e^{-\pi i/4}\sqrt{|\sigma|}$ will be taken.

This set of approximate solutions of equation (7), valid outside the turning point region, constitute the outer solution. Inside the turning point region this approximation breaks down. The approximation is invalid here, because neglected terms of equation (7) are now dominant, and another approximate equation is to be used. This will give us the inner or boundary layer solution. To determine the unknown constants (here $\mathscr{R}$ and $\mathscr{T}$), the inner and outer solutions are asymptotically matched.

For the matching it is necessary to determine the asymptotic behaviour of the outer solution in the limit $X \to X_t$, and the boundary layer thickness (i.e. the appropriate local coordinate).

From the limiting behaviour of the outer solution in the turning point region (see below), we can estimate the order of magnitude of the solution. From a balance of terms in the differential equation (7) it transpires that the turning-point boundary layer is of thickness $X - X_t = O(\varepsilon^{2/3})$, leading to a boundary layer variable ξ given by

$$X = X_t + \varepsilon^{2/3} \lambda^{-1} \xi \tag{58}$$

where λ is introduced for notational convenience later, and is given by

$$\lambda^3 = \frac{2\omega^2 C_{0t}^2}{(C_{0t}^2 - U_{0t}^2)^2} \left(\frac{C_{0t} C_{0t}' - U_{0t} U_{0t}'}{C_{0t}^2 - U_{0t}^2} + \frac{\alpha_t'}{\alpha_t}\right). \tag{59}$$
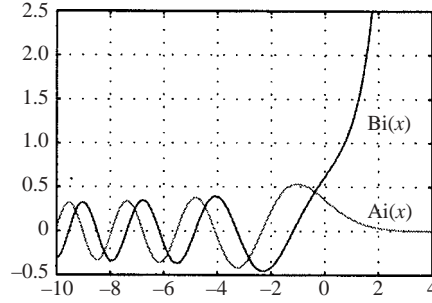
FIGURE 4. Airy functions.

By assumption $\lambda = O(1)$. Since for $\varepsilon \to 0$

$$\sigma^2(X) = \sigma^2\left(X_t + \varepsilon^{2/3}\lambda^{-1}\xi\right) = -2\varepsilon^{2/3}\left(\frac{C_{0t}C'_{0t} - U_{0t}U'_{0t}}{C^2_{0t} - U^2_{0t}} + \frac{\alpha'_t}{\alpha_t}\right)\lambda^{-1}\xi + O\left(\varepsilon^{4/3}\xi^2\right), \tag{60}$$

we have

$$\frac{1}{\varepsilon}\int_{X_t}^{X}\frac{\omega C_0 \sigma}{C^2_0 - U^2_0}\,\mathrm{d}X' = \begin{cases} -\frac{2}{3}(-\xi)^{3/2} = -\zeta & \text{if } \xi < 0 \\ -\mathrm{i}\frac{2}{3}\xi^{3/2} = -\mathrm{i}\zeta & \text{if } \xi > 0 \end{cases} \tag{61}$$

where we have introduced $\zeta = \frac{2}{3}|\xi|^{3/2}$. The limiting behaviour for $X \uparrow X_t$ is now given by

$$\phi \simeq \frac{n_t}{\varepsilon^{1/6}(-\xi)^{1/4}}\left(\frac{\omega C_{0t}}{\lambda\left(C^2_{0t} - U^2_{0t}\right)}\right)^{1/2}\psi(r,\theta;X_t)(\mathrm{e}^{\mathrm{i}\zeta} + \mathscr{R}\,\mathrm{e}^{-\mathrm{i}\zeta}), \tag{62}$$

while for $X \downarrow X_t$ it is given by

$$\phi \simeq \mathscr{T}\frac{n_t}{\varepsilon^{1/6}\xi^{1/4}}\left(\frac{\omega C_{0t}}{\lambda\left(C^2_{0t} - U^2_{0t}\right)}\right)^{1/2}\mathrm{e}^{\pi \mathrm{i}/4}\,\psi(r,\theta;X_t)\mathrm{e}^{-\zeta}. \tag{63}$$

Since the boundary layer is relatively thin, also compared to the radial coordinate, the behaviour of the incident mode remains quite unaffected in the radial direction, and we can assume in the turning point region

$$\phi(x,r,\theta) = \chi(\xi)\psi(r,\theta;X)\exp\left(\frac{\mathrm{i}}{\varepsilon}\int_{X_t}^{X}\frac{\omega U_0}{C^2_0 - U^2_0}\mathrm{d}X'\right), \tag{64}$$

where $X = X_t + \varepsilon^{2/3}\lambda^{-1}\xi$ and $\xi = O(1)$. Substitution in equation (7), and using the defining equation (33) of ψ, we arrive at

$$\varepsilon^{2/3}\left(1 - \frac{U^2_{0t}}{C^2_{0t}}\right)\lambda^2\psi(r,\theta;X_t)(\chi'' - \xi\chi) = O(\varepsilon). \tag{65}$$

So to leading order we have Airy's equation

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\xi^2} - \xi\chi = 0. \tag{66}$$

This has the general solution (figure 4)

$$\chi(\xi) = a\,\mathrm{Ai}(\xi) + \ell\,\mathrm{Bi}(\xi), \tag{67}$$

where a and ℓ, parallel to $\mathscr{R}$ and $\mathscr{T}$, are to be determined from matching. Using the asymptotic expressions (B 2b, d) in Appendix B for Airy functions, we find that for ξ large with $1 \ll \xi \ll \varepsilon^{-2/3}$, equation (63) matches the inner solution if

$$\frac{a}{2\sqrt{\pi}\xi^{1/4}}e^{-\zeta} + \frac{\ell}{\sqrt{\pi}\xi^{1/4}}e^{\zeta} \sim \mathscr{T}\frac{n_t}{\varepsilon^{1/6}\xi^{1/4}}e^{\pi i/4}\left(\frac{\omega C_{0t}}{\lambda(C_{0t}^2 - U_{0t}^2)}\right)^{1/2}e^{-\zeta}. \tag{68}$$

Since $e^{\zeta} \to \infty$, we can only have $\ell = 0$, and thus

$$a = \frac{2n_t\sqrt{\pi}}{\varepsilon^{1/6}}\left(\frac{\omega C_{0t}}{\lambda(C_{0t}^2 - U_{0t}^2)}\right)^{1/2}e^{\pi i/4}\mathscr{T}. \tag{69}$$

If $-\xi$ is large with $1 \ll -\xi \ll \varepsilon^{-2/3}$ we use the asymptotic expression (B 2a), and find that equation (62) matches the inner solution if

$$a\frac{\cos\left(\zeta - \frac{1}{4}\pi\right)}{\sqrt{\pi}(-\xi)^{1/4}} \sim \frac{n_t}{\varepsilon^{1/6}(-\xi)^{1/4}}\left(\frac{\omega C_{0t}}{\lambda(C_{0t}^2 - U_{0t}^2)}\right)^{1/2}(e^{i\zeta} + \mathscr{R}e^{-i\zeta}), \tag{70}$$

which is equivalent to the following identity in variable ζ:

$$\mathscr{T}e^{i\zeta} + \mathscr{T}ie^{-i\zeta} \equiv e^{i\zeta} + \mathscr{R}e^{-i\zeta}. \tag{71}$$

This is true for any ζ if

$$T = 1, \qquad R = i. \tag{72}$$

The amplitudes of these reflection and transmission coefficients could of course be guessed by conservation-of-energy arguments. This is not the case with the phase. It appears that the wave reflects with a phase change of $\frac{1}{2}\pi$, while the transmission is without phase change.

7. Conclusions

The problem of sound propagation in slowly varying lined ducts of arbitrary cross-section with isentropic irrotational mean flow is solved in principle. No attempt has been made yet to illustrate the results with numerical examples, because the corresponding eigenvalue problem in a cross-section is not straightforward. Further work is underway to implement the present results numerically.

The present generalization gives much insight into previous results for circular and elliptic ducts (which are special cases of the present results), because the form of the solution is seen to become very simple through the normalization of the eigenfunctions used.

An interesting phenomenon of mode propagating in hard-walled ducts of varying cross-section is their change from propagating (cut-on) to exponentially decaying (cut-off) at a so-called turning point. The present multiple-scales solution allows the analysis of this turning point behaviour. The results are quite similar to those for the circular duct case. It seems possible to extend our analysis to the quasi-turning point behaviour in ducts with lined walls reported by Ovenden (2002).

This work was carried out in the context of the "TurboNoiseCFD" project of the European Union's 5th Framework "Growth" Programme (EU Technical Officer: Per Kruppa, Co-ordinator: Brian Tester).

Appendix A. Proof of Lemma 1

For any sufficiently smooth vector field $\boldsymbol{f}$, with $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ at the tube surface $r = T(x, \theta)$, we have

$$\int_{\partial \mathscr{A}} [\nabla \cdot \boldsymbol{f} - \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{f})] \sqrt{1 + \frac{T^2 T_x^2}{T^2 + T_\theta^2}} \, \mathrm{d}\ell = \frac{\mathrm{d}}{\mathrm{d}x} \int_{\partial \mathscr{A}} (\boldsymbol{f} \cdot \boldsymbol{v}) \, \mathrm{d}\ell,$$

where $\boldsymbol{v} = \boldsymbol{n} \times \boldsymbol{e}_\ell$ is the unit vector normal to $\partial \mathscr{A}$, tangential to the surface and pointing in the positive x-direction.

Proof. We will use the powerful apparatus of tensor calculus for curvilinear coordinates. See for example Sokolnikoff (1951).

Assume that any point $\boldsymbol{x}$ and unit normal vector $\boldsymbol{n}$ on the surface $r = T(x, \theta)$ are parameterized by $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ and $\boldsymbol{n} = \boldsymbol{n}(u^1, u^2)$, where $u^1 = x$ and $u^2 = \theta$. Introduce in the neighbourhood of the surface the curvilinear coordinate system (u^1, u^2, u^3) by the mapping

$$(u^1, u^2, u^3) \mapsto \boldsymbol{x}(u^1, u^2) + u^3 \boldsymbol{n}(u^1, u^2).$$

This generates base vectors $\boldsymbol{a}_1 = \boldsymbol{x}_{u^1} + u^3 \boldsymbol{n}_{u^1}$, $\boldsymbol{a}_2 = \boldsymbol{x}_{u^2} + u^3 \boldsymbol{n}_{u^2}$, $\boldsymbol{a}_3 = \boldsymbol{n}$ and the corresponding metric tensor $(g_{ij}) = (\boldsymbol{a}_i \cdot \boldsymbol{a}_j)$ and its inverse (g^{ij}). Since $|\boldsymbol{n}| = 1$ and $\boldsymbol{x}_{u^1}$ is tangent to the surface, both $\boldsymbol{x}_{u^1} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{n}_{u^1} \cdot \boldsymbol{n} = 0$. (The same is true for u^2.) As a result, the third column and row of (g_{ij}) and of (g^{ij}) are of the form $[0, 0, 1]$. We introduce the Christoffel symbols

$$\begin{Bmatrix} m \\ i \, j \end{Bmatrix} = \sum_{k=1}^{3} \tfrac{1}{2} g^{mk} \left[\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right].$$

From the structure of (g_{ij}) and (g^{ij}) it follows immediately that for any j

$$\begin{Bmatrix} 3 \\ 3 \, j \end{Bmatrix} = 0.$$

Consider the vector field $\boldsymbol{f} = f^1 \boldsymbol{a}_1 + f^2 \boldsymbol{a}_2 + f^3 \boldsymbol{a}_3$. Taking twice the inner product with $\boldsymbol{n}$ of the covariant derivatives

$$(\nabla \boldsymbol{f})^i_j = \frac{\partial f^i}{\partial u^j} + \sum_{k=1}^{3} \begin{Bmatrix} i \\ j \, k \end{Bmatrix} f^k$$

now yields

$$\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{f}) = (\nabla \boldsymbol{f})^3_3 = \frac{\partial f^3}{\partial u^3} + \sum_{k=1}^{3} \begin{Bmatrix} 3 \\ 3 \, k \end{Bmatrix} f^k = \frac{\partial f^3}{\partial u^3}.$$

At the surface, where $\boldsymbol{f} \cdot \boldsymbol{n} = f^3 = 0$, the divergence can be written as follows:

$$\nabla \cdot \boldsymbol{f} = \sum_{i=1}^{3} \left(\frac{\partial f^i}{\partial u^i} + \sum_{k=1}^{3} \begin{Bmatrix} i \\ i \, k \end{Bmatrix} f^k \right) = \sum_{i=1}^{2} \left(\frac{\partial f^i}{\partial u^i} + \sum_{k=1}^{2} \begin{Bmatrix} i \\ i \, k \end{Bmatrix} f^k \right) + \frac{\partial f^3}{\partial u^3} = \nabla_T \cdot \boldsymbol{f}_T + \frac{\partial f^3}{\partial u^3}$$

where $\boldsymbol{f}_T = f^1\boldsymbol{a}_1 + f^2\boldsymbol{a}_2$ and $\nabla_T \cdot \boldsymbol{f}_T$ denotes the surface divergence of $\boldsymbol{f}$. As a result we have the remarkable identity

$$\nabla \cdot \boldsymbol{f} - \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{f}) = \nabla_T \cdot \boldsymbol{f}_T.$$

This divergence suggests applying Gauss' theorem along the surface considered. Consider a strip $\mathscr{S}$ wrapped around the surface between two cross-sections at x and $x + \Delta x$, with circumferences denoted by $\partial\mathscr{A}_0$ and $\partial\mathscr{A}_1$. Note that $-\boldsymbol{v}$ and $\boldsymbol{v}$ are the respective outward normal unit vectors of the boundary of $\mathscr{S}$.

With Gauss' divergence theorem we have then

$$\iint_{\mathscr{S}} (\nabla_T \cdot \boldsymbol{f}_T)\,\mathrm{d}\sigma = \int_{\partial\mathscr{A}_1} (\boldsymbol{f} \cdot \boldsymbol{v})\,\mathrm{d}\ell - \int_{\partial\mathscr{A}_0} (\boldsymbol{f} \cdot \boldsymbol{v})\,\mathrm{d}\ell.$$

It may be noted that this result could also be obtained by identifying $\nabla \cdot \boldsymbol{f} - \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla \boldsymbol{f}) = \boldsymbol{n} \cdot (\nabla \times (\boldsymbol{n} \times \boldsymbol{f}))$ (following Möhring 2001 and Eversman 2001), and then applying Stokes theorem.

We rewrite the first integral in the coordinates $u^1 = x'$ and $u^2 = \theta$ by

$$\mathrm{d}\sigma = \left|\frac{\partial\boldsymbol{x}}{\partial\theta} \times \frac{\partial\boldsymbol{x}}{\partial x}\right| \mathrm{d}\theta\,\mathrm{d}x' = \sqrt{T^2 + T_\theta^2 + T^2 T_x^2}\,\mathrm{d}\theta\,\mathrm{d}x'.$$

If we let Δx tend to zero and change $\mathrm{d}\theta = |\boldsymbol{x}_\theta|^{-1}\mathrm{d}\ell = (T^2 + T_\theta^2)^{-\frac{1}{2}}\mathrm{d}\ell$, we obtain

$$\iint_{\mathscr{S}} (\nabla_T \cdot \boldsymbol{f}_T)\,\mathrm{d}\sigma = \int_x^{x+\Delta x} \int_0^{2\pi} (\nabla_T \cdot \boldsymbol{f}_T)\sqrt{T^2 + T_\theta^2 + T^2 T_x^2}\,\mathrm{d}\theta\,\mathrm{d}x'$$

$$\approx \Delta x \int_0^{2\pi} (\nabla_T \cdot \boldsymbol{f}_T)\sqrt{T^2 + T_\theta^2 + T^2 T_x^2}\,\mathrm{d}\theta = \Delta x \int_{\partial\mathscr{A}} (\nabla_T \cdot \boldsymbol{f}_T)\sqrt{1 + \frac{T^2 T_x^2}{T^2 + T_\theta^2}}\,\mathrm{d}\ell.$$

Divide by Δx, and the result follows by taking the limit for $\Delta x \to 0$ and using continuity of the integrand in x. $\qquad\square$

Appendix B. Airy functions

Related to Bessel functions of order $\frac{1}{3}$ are the Airy functions Ai and Bi, solution of

$$y'' - xy = 0, \tag{B 1}$$

(Abramowitz & Stegun 1964) with the following asymptotic behaviour (introduce $\zeta = \frac{2}{3}|x|^{3/2}$):

$$\mathrm{Ai}(x) \simeq \frac{\cos\left(\zeta - \frac{1}{4}\pi\right)}{\sqrt{\pi}|x|^{1/4}} \quad (x \to -\infty) \tag{B 2a}$$

$$\simeq \frac{\mathrm{e}^{-\zeta}}{2\sqrt{\pi}x^{1/4}} \quad (x \to \infty), \tag{B 2b}$$

$$\mathrm{Bi}(x) \simeq \frac{\cos\left(\zeta + \frac{1}{4}\pi\right)}{\sqrt{\pi}|x|^{1/4}} \quad (x \to -\infty) \tag{B 2c}$$

$$\simeq \frac{\mathrm{e}^{\zeta}}{\sqrt{\pi}x^{1/4}} \quad (x \to \infty). \tag{B 2d}$$

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